COMPUTATION OF A LOWER BOUND FOR A VEHICLE ROUTING PROBLEM WITH MOTION CONSTRAINTS

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ABSTRACT

Given a set of targets that need to be monitored and a vehicle, we consider a combinatorial motion planning problem where the objective is to find a path for the vehicle such that each target is visited at least once by the vehicle, the path satisfies the motion constraints of the vehicle and the length of the path is a minimum. This is an NP-hard problem and currently, there are no algorithms that can find an optimal solution to this problem. In this article, we model the motion of the vehicle as a Dubins car and develop a method that can provide tight lower bounds to the motion planning problem. We accomplish this by relaxing the constraints corresponding to the angle of approach at each of the targets and then penalizing them whenever they are violated. The solution to the Lagrangian relaxation gives a lower bound, and this lower bound is maximized over the penalty variables using subgradient optimization. The proposed method is the first of its kind for finding tight lower bounds for combinatorial motion planning problems and can be extended to similar problems with more general motion constraints.

1 INTRODUCTION

For a given set of targets, the objective of the Traveling Salesman Problem (TSP) is to find a tour of minimum distance that starts from one of the targets and visit every other target at least once and returns to the starting target. This problem was studied extensively by several authors [1], [2] and [3]. The objective of the problem considered in this article is to find a path for the vehicle such that each target is visited at least once by the vehicle, the path satisfies the motion constraints of the vehicle and the length of the path is a minimum. If the motion constraints are relaxed, this path planning problem reduces to the Euclidean TSP. The vehicle can be an Unmanned Aerial Vehicle (UAV) where the motion of the vehicle has to satisfy a given set of constraints. The motion constraint we consider here is that the yaw rate of the vehicle at any instance along its path is upper bounded by a constant. Problems of this type arise naturally in military and civil applications where UAVs are used for border surveillance, forest fire monitoring, weather monitoring etc. Given any two targets i, j, the motion constraints for the vehicle to travel from target i to target j are as follows:

$$\dot{\zeta} = \cos\theta, \quad \dot{\eta} = \sin\theta, \quad \dot{\theta} = u, \quad |u| \leq \Omega, \tag{1}$$
$$\zeta(0) = x_i, \quad \eta(0) = y_i, \quad \theta(0) = \theta_i, \tag{2}$$
$$\zeta(t_f) = x_j, \quad \eta(t_f) = y_j, \quad \theta(t_f) = \theta_j, \tag{3}$$

where $(\zeta(t), \eta(t))$ is the position of the UAV at time t, and (x_i, y_i) and (x_j, y_j) are the coordinates of the target i and j in the two dimensional plane respectively. θ_i and θ_j represent a given heading angle of the vehicle at targets i and j. $\dot{\theta}$ is the yaw rate and is bounded from above by Ω. The minimum turning radius of the vehicle is a function of Ω, was assumed to be 1 without loss of generality. t_f is the time when the vehicle reaches its final configuration at (x_j, y_j, θ_j). For a given set of targets $N = (1, 2, ...n)$, i.e. coordinates of the n nodes, the vehicle has to start from one of the targets, visit all the remaining targets and return to the first target while satisfying all the motion constraints (1-3). If the vehicle reaches a target j with a heading angle θ_j, the heading angle while leaving the target j should also be equal to θ_j. The objective of the path planning problem is to minimize the total distance travelled by the vehicle while visiting all the targets. We refer to the vehicle with the motion constraints specified in (1-3) as the Dubins vehicle, and the path planning problem as the Dubins Traveling Salesman Problem (DTSP). If we relax the motion constraints of the vehicle, the shortest distance between any two targets is the Euclidean distance. The corresponding routing problem is an Euclidean Traveling Salesman Problem (ETSP) and solving it provides a lower bound for the DTSP. The routing problems of this genre were earlier studied in [4], [5], [6], [7] and [8]. [4] and [5] provides an approximate solution guaranteed to be within a constant factor of optimum, [6] describes a bead-tiling algorithm with asymptotic guarantees. In [7], a two step approach is prescribed to solve a Multi Depot Multiple TSP. The sequence of cities to be visited is solved as a combinatorial problem and the heading angles at each target are computed using dynamic programming. The work in [8] deals with a Heterogeneous Multi Depot Multiple UAV Routing Problem (HMDMURP) which is in turn transformed into a standard Asymmetric TSP and solved using the Lin-Kernighan Heuristic (LKH).

Dubins TSP is hard to solve as we have to find the optimal heading angles and the optimal sequence of targets to be visited. Currently, there are heuristics and approximation algorithms to solve this DTSP in polynomial time. However, there are no algorithms that can either find an optimal solution or a good lower bound for the Dubins TSP. Lower bounds are important because they can be used to corroborate the quality of the solutions produced by the heuristics or the approximation algorithms. In this article, we are interested in finding a lower bound for the Dubins TSP which is tighter than the existing lower bound provided by the Euclidean TSP. A tighter lower bound is also useful in branch and bound procedures for discarding a set of solutions that are guaranteed not to contain the optimal solution.

We compute the lower bounds for the Dubins TSP using Lagrangian relaxation. A Lagrangian relaxation is obtained by removing some of the constraints in the Dubins TSP and penalizing them in the objective whenever it is violated. Using the weak duality theorem, it is well known that this Lagrangian relaxation is a lower bound to the primal problem. In this article, the objective function of the Lagrangian relaxation is posed as an asymmetric TSP where the cost of traveling each edge is computed by solving a variational problem. This asymmetric TSP is solved using the Lin-Kernighan heuristic (LKH) [9] which is one of the best known heuristics for the TSP in the literature. For any given set of dual variables, the solution of this relaxation gives a lower bound to the Dubins TSP. Therefore, subgradient optimization techniques are used in order to obtain the best lower bound.

This paper is organized in the following format. The problem statement is presented in section 2. In section 3, computation of a lower bound from Lagrangian relaxation and subgradient optimization is explained. Numerical results and comparison with a transformation technique from [8] are presented in sections 4,5.

2 PROBLEM FORMULATION

Let N be a set of n targets and E be the set of edges between the n targets. (x_i, y_i) are the x and y coordinates of target i on an $x-y$ plane, θ_i is the heading angle of the Dubins vehicle at target i. Let T be the set of all tours on the graph given by the locations of the targets and Θ_i be the set of allowed headings for the i^{th} target. Let $\mathbf{x}$ be the incidence matrix of decision variables, whose entry in the i^{th} row and j^{th} column is x_{ij}. We will say that $\mathbf{x} \in T$, if the incidence matrix corresponds to a tour. The traveling salesman problem for a Dubins vehicle can be stated as following:

$$J^* = \min_{\theta_i, x_{ij}} \sum_{(i,j)\in E} d_{ij}(\theta_i, \theta_j) x_{ij}. \tag{4}$$

subject to:

$$\mathbf{x} \in T, \quad \theta_i \in \Theta_i, \ i \in N, \tag{5}$$

where $d_{ij}(\theta_i, \theta_j)$ is the length of the shortest path of the Dubins vehicle starting from target i at (x_i, y_i) with a heading θ_i, going to target j at (x_j, y_j) with a heading θ_j. d_{ij} can be expressed in terms of the dynamics of the Dubins vehicle as:

$$d_{ij}(\theta_i, \theta_j) = \min_{u_{ij}} \quad t_{ij}, \tag{6}$$

Subject to

$$\dot{\zeta}_{ij} = \cos\theta_{ij}, \quad \dot{\eta}_{ij} = \sin\theta_{ij}, \quad \dot{\theta}_{ij} = u_{ij}, \quad |u_{ij}| \leq \Omega, \tag{7}$$
$$\zeta_{ij}(0) = x_i, \quad \eta_{ij}(0) = y_i, \tag{8}$$
$$\zeta_{ij}(t_{ij}) = x_j, \quad \eta_{ij}(t_{ij}) = y_j, \tag{9}$$
$$\theta_{ij}(0) = \theta_i, \quad \theta_{ij}(t_{ij}) = \theta_j, \tag{10}$$

Here ζ_{ij} and η_{ij} are the displacement variables of the Dubins vehicle in x and y directions, $\theta_{ij}(t)$ is the angle made by the tangent to the path of the vehicle at time t with the x-axis and t_{ij} is the time at which the vehicle reaches target j. Suppose the final tour contains the edges (i,j) and (j,k), the final heading of the vehicle traveling from target i to target j should be equal to the initial heading while traveling from target j to target k. That is at each target the heading of arrival is same as the heading of departure. We do not know which target precedes which in the final tour, but we know that there is only one edge going towards target j and only one edge going out of it. So, we can make use of the binary variables x_{ij} and sum over the index $i \in N$ to write the heading angle constraint as:

$$\sum_{i:(i,j)\in E} \theta_{ij}(t_{ij})x_{ij} - \sum_{k:(j,k)\in E} \theta_{jk}(0)x_{jk} = 0, \quad \forall j \in N. \tag{11}$$

Equations (10) and (11) are in general a difficult constraints to deal with. The domain of θ_{ij} is cylindrical and one has to identify 0 and 2π as one and the same. Instead we will pose these constraints in terms of sines and cosines of the angles θ_{ij} as shown below:

$$\cos\theta_{ij}(0) = \cos\theta_i, \quad \sin\theta_{ij}(0) = \sin\theta_i, \tag{12}$$

$$\cos\theta_{ij}(t_{ij}) = \cos\theta_j, \quad \sin\theta_{ij}(t_{ij}) = \sin\theta_j. \tag{13}$$

The constraint on the heading angles (11) at each target j can be written in terms of sines and cosines as:

$$\sum_{i:(i,j)\in E} \cos\theta_{ij}(t_{ij})x_{ij} - \sum_{k:(j,k)\in E} \cos\theta_{jk}(0)x_{jk} = 0, \tag{14}$$

$$\sum_{i:(i,j)\in E} \sin\theta_{ij}(t_{ij})x_{ij} - \sum_{k:(j,k)\in E} \sin\theta_{jk}(0)x_{jk} = 0, \quad \forall j \in N. \tag{15}$$

3 Main Result

3.1 Computation of lower bound

The solution of a Lagrangian relaxation problem obtained by penalizing some of the constraints readily gives a lower bound for the original minimization problem. A lower bound for the minimization problem defined by equations (4) to (15) can be computed by penalizing the objective function(4) with the constraints (14) and (15) using dual variables $\Pi = [\alpha_j, \beta_j]$, $\alpha_j, \beta_j \in \Re$, $j = 1$ to n.

$$\begin{aligned} L(\Pi) = \min_{\theta_i, x_{ij}} & \sum_{(i,j)\in E} d_{ij}(\theta_i,\theta_j)x_{ij} \\ & - \sum_{j\in N}\alpha_j \left[\sum_{i:(i,j)\in E}\cos\theta_{ij}(t_{ij})x_{ij} - \sum_{k:(j,k)\in E}\cos\theta_{jk}(0)x_{jk}\right] \\ & - \sum_{j\in N}\beta_j \left[\sum_{i:(i,j)\in E}\sin\theta_{ij}(t_{ij})x_{ij} - \sum_{k:(j,k)\in E}\sin\theta_{jk}(0)x_{jk}\right]. \end{aligned} \tag{16}$$

Here the summations are expanded and the terms are rearranged as shown in equation (17). Also $\cos\theta_{ij}(t_{ij})$ and $\cos\theta_{jk}(0)$ can be replaced with $\cos\theta_j$, and similarly $\sin\theta_{ij}(t_{ij})$ and $\sin\theta_{jk}(0)$ can be replaced with $\sin\theta_j$, and thus equation (16) reduces to

$$L(\Pi) = \min_{\theta_i, x_{ij}} \sum_{(i,j)\in E} [d_{ij}(\theta_i,\theta_j) - \alpha_j\cos\theta_j - \beta_j\sin\theta_j + \alpha_i\cos\theta_i + \beta_i\sin\theta_i]x_{ij}. \tag{17}$$

$$\geq \min_{x_{ij}} \sum_{(i,j)\in E} \min_{(\theta_i,\theta_j)} [d_{ij}(\theta_i,\theta_j) - \alpha_j\cos\theta_j - \beta_j\sin\theta_j + \alpha_i\cos\theta_i + \beta_i\sin\theta_i]x_{ij}. \tag{18}$$

Let us call the objective function in (18) $J(\Pi)$.

$$J(\Pi) = \min_{x_{ij}} \sum_{(i,j)\in E} \min_{(\theta_i,\theta_j)} [d_{ij}(\theta_i,\theta_j) - \alpha_j\cos\theta_j - \beta_j\sin\theta_j + \alpha_i\cos\theta_i + \beta_i\sin\theta_i]x_{ij}. \tag{19}$$

Theorem: For any given Π, the solution of the minimization problem with the objective function (19) and constraints (5) is a lower bound to the Dubins TSP (4) to (5) and (14) to (15).
Proof: Clearly $L(\Pi)$ in (16) is the Lagrangian relaxation of the primal problem (4). The weak duality theorem states that for a minimization problem, the solution of the Lagrangian relaxation is less than any primal feasible solution. Therefore, $(L(\Pi))$ is a lower bound to the DTSP in (4). One can see that $J(\Pi)$ is always less than $L(\Pi)$ from equation (18). Thus for any given Π, the solution to (19) is a lower bound to the Dubins Traveling Salesman Problem.

Consider the following variational problem

$$v_{ij}(\alpha_i,\alpha_j,\beta_i,\beta_j) = \min_{\theta_i,\theta_j\in\Theta} d_{ij}(\theta_i,\theta_j) - \alpha_j\cos\theta_j - \beta_j\sin\theta_j + \alpha_i\cos\theta_i + \beta_i\sin\theta_i. \tag{20}$$

where $d_{ij}(\theta_i,\theta_j)$ is given by equations (6) to (9) and (12) to (13). $d_{ij}(\theta_i,\theta_j)$ is the minimum Dubins distance from the configuration (x_i,y_i,θ_i) to (x_j,y_j,θ_j) and can be calculated using the result from Dubins [10]. Given the values of $\alpha_i,\alpha_j,\beta_i,\beta_j$, one can compute v_{ij} as prescribed here. Discretize the allowable values of the heading angle and obtain a discrete set of heading angles $\Theta_d = \{\theta_1, \theta_2, ...\theta_m\}$. We assume that this discrete set of heading angles is the same for all the targets, i.e $\Theta_i = \Theta_d, \forall i \in N$. Therefore, for every pair of $\theta_i, \theta_j \in \Theta_d$, the value of $d_{ij}(\theta_i, \theta_j) - \alpha_j \cos\theta_j - \beta_j \sin\theta_j + \alpha_i \cos\theta_i + \beta_i \sin\theta_i$ can be calculated. The minimum of all these values each corresponding to a pair of $\theta_i, \theta_j \in \Theta_d$ is v_{ij}. Given $\Pi = (\alpha_1, ...\alpha_n, \beta_1, ...\beta_n)$, v_{ij} is computed for every edge $(i, j) \in E$. Now the minimization problem(19) can be stated as following:

$$J(\Pi) = \min_{x_{ij}, i, j \in N} \sum_{(i,j)\in E} v_{ij}(\alpha_i, \alpha_j, \beta_i, \beta_j) x_{ij}, \tag{21}$$

subject to

$$\mathbf{x} \in T. \tag{22}$$

This is an asymmetric traveling salesman problem where weight of each edge is v_{ij}. Once v_{ij} is computed for all $(i, j) \in E$, the ATSP can be solved using LKH heuristic, which is a lower bound to the DTSP. The computation of a lower bound for a given set of penalizing variables(Π) is summerized below:

1. Select the dual variables $\Pi = (\alpha_1, ..\alpha_n, \beta_1, ..\beta_n)$.
2. Use Π^k to formulate the Lagrangian relaxation and the variational problem(20).
3. Solve the variational problem (20) for every $(i, j) \in E$.
4. Solve the asymmetric Traveling Salesman Problem (21) using LKH heuristic.

Since J^* is greater than $J(\Pi)$ for any Π, one can maximize $J(\Pi)$ over the dual variables Π to compute a tighter lower bound.

$$J^* \geq \max_{\Pi} J(\Pi). \tag{23}$$

Since $J(\Pi)$ is a combination of finite number of linear functions, it is concave in Π. One technique that works well to maximize this kind of problems is subgradient optimization.

3.2 Subgradient Optimization

Subgradient optimization is an iterative procedure where a set of dual variables Π^{k+1} are computed after each iteration k using Π^k from previous iteration. The procedure starts with an initial vector Π^0, and after each iteration k, a new vector Π^{k+1} is computed by taking a step along the subgradient direction:

$$\Pi^{k+1} = \Pi^k + \delta^k s^k, \tag{24}$$

where s^k is the subgradient direction and δ^k is the step size along the subgradient. We will explain this in the context of the following minimization problem.

$$J = \min_{x\in X} f(x),$$

subjected to:

$$Ax = b.$$

The Lagrangian relaxation obtained by relaxing the constraints $Ax = b$ is:

$$L = \min_{x\in X} f(x) + \lambda(b - Ax),$$

where $\lambda = [\lambda_1, ...\lambda_n]$ are the dual variables penalizing the constraint $Ax = b$. In general a subgradient direction for a problem in this form can be given by [11] $s^k = b - Ax^k$, where x^k is the solution of the dual problem in the iteration k. One can select a constant step size or a diminishing step size or a Polyak's step size. The convexity of the Lagrangian function guarantees the convergence of this subgradient optimization.

3.2.1 Implementation details of the subgradient optimization technique:

An important part of the subgradient optimization technique is to find a direction of subgradient and step size at each iteration k. Since constraints (14) and (15) are relaxed, one can chose the following as the subgradient:

$$s^k = \begin{bmatrix} \sum_{i:(i,j)\in E} \cos\theta^k_{ij}(t_{ij}) x^k_{ij} - \sum_{k:(j,k)\in E} \cos\theta^k_{jk}(0) x^k_{jk} \\ \sum_{i:(i,j)\in E} \sin\theta^k_{ij}(t_{ij}) x^k_{ij} - \sum_{k:(j,k)\in E} \sin\theta^k_{jk}(0) x^k_{jk} \end{bmatrix}. \tag{25}$$

Here x^k_{ij} is the solution of (21) in the iteration k and the values of θ^k_{ij} are from the solution of the problem defined by equation (20) in iteration k. The step size δ^k at iteration k is computed using Polyak's rule:

$$\delta^k = \gamma^k \frac{J^u - J(\Pi^k)}{||s^k||}, \tag{26}$$

where J^u is any known upper bound to the DTSP, which can be calculated using the result from [8] and γ^k is a constant initially and reduces by a constant factor after a specified number of iterations. This iterative procedure is outlined as below:

1. Initialize $k = 0$, $\Pi^k = \Pi^0$, $\gamma^k = \gamma^0$.
2. Compute v_{ij} as shown in (20), for all $(i, j) \in E$.
3. Solve the asymmetric TSP (21) to (22) using LKH routine.
4. If $\frac{J^k - J^{k-1}}{J^{k-1}} \leq \varepsilon$ or $k = k_{max}$, go to 6.
5. Compute $\Pi^{k+1} = \Pi^k + \delta^k s^k$, where s^k and δ^k are given by equations (25) and (26) respectively, set $k = k + 1$ and go to 2.

6. Stop the iterative procedure.

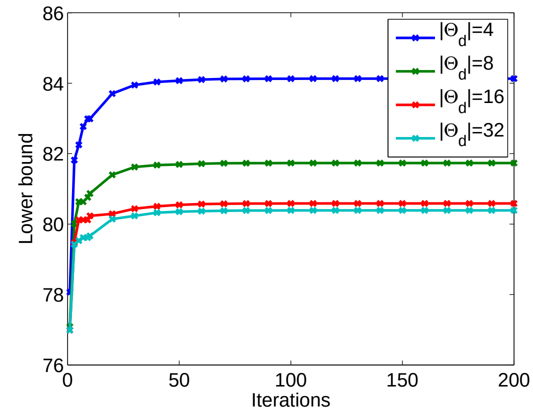
In step(4), $\varepsilon > 0$ is a small number specified to check the convergence and k_{max} is the maximum number of iterations allowed. γ^k was chosen to be 0.05 initially and is reduced by a factor of 2 after every 10 iterations. In the step(3), we used the readily available LKH implementation, which could take only integral values for the edge weights as the input. But the value of v_{ij} computed in (20) might be any positive real number. So, we multiplied these values with 1000 and rounded off to the nearest integers so that they can be used as input to the LKH routine. The output from the LKH routine is divided by 1000 to get the actual length of the tour. The convergence of subgradient optimization is quite slow after around 20 iterations and it shows a zigzagging behaviour. We chose the best value of $J(\Pi)$ after specified number of iterations.

4 One in a set Transformation

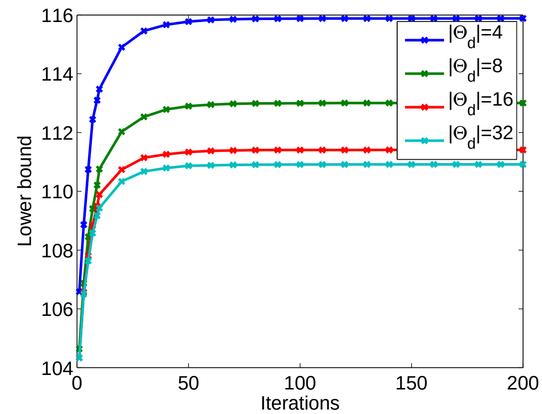
To corroborate the performance of the proposed technique, we also solve the DTSP using the method in Oberlin et. al [8]. In [8], the authors solve the DTSP (where the choice of the heading angle at each target is restricted to a discrete set) by transforming the DTSP into an asymmetric TSP. They replicate each target m times such that each of the m replications correspond to a possible heading angle. Now, the DTSP is posed as a problem of finding a subtour for the vehicle such that exactly one copy of each target is visited once and the total distance traveled by the vehicle is a minimum. This problem is then transformed into an asymmetric TSP using the method presented by Noon and Bean in [12]. One can solve this asymmetric TSP using LKH heuristic. The disadvantage of this transformation is that the problem size increases with the number of discretizations of the heading angle and hence, can become computationally more difficult to solve. This algorithm readily gives an approximate solution to the DTSP, which is an upper bound to the optimum. This transformation method is useful because we can use the upper bound provided by this transformation method for calculating the Polyak's step size in the subgradient optimization. This method is also useful because it can provide a lower bound which in turn can be used to compare with the lower bounds obtained using the method proposed in this article.

5 NUMERICAL RESULTS

The plots of the lower bound versus the number of iterations of subgradient optimization for two problem instances with 20 targets and 40 targets are shown in Figure 1(a) and Figure 1(b). Each of the figures also show the plots for different discretizations(Θ_d) of the heading angle. As the size of the set Θ_d is increased, the lower bound reduces. This is because the solution of the variational problem (20) gives a better minimum with more discretizations of the heading angle θ. One can also infer from Figure 1(a) and Figure 1(b) that the final value of lower



(a) Instance with 20 targets



(b) Instance with 40 targets

Figure 1. CONVERGENCE OF SUBGRADIENT OPTIMIZATION

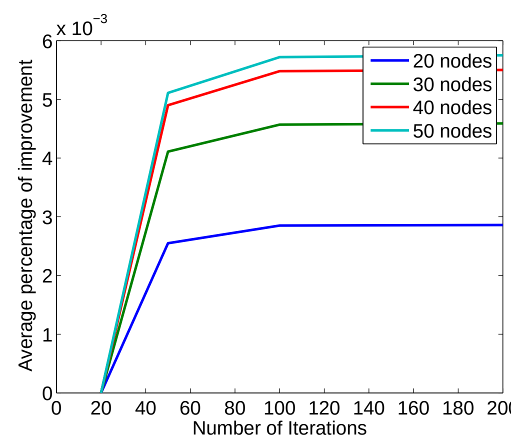


Figure 2. AVERAGE OF THE PERCENTAGE OF IMPROVEMENT COMPARED TO 20 ITERATIONS, WITH 32 DISCRETIZATIONS

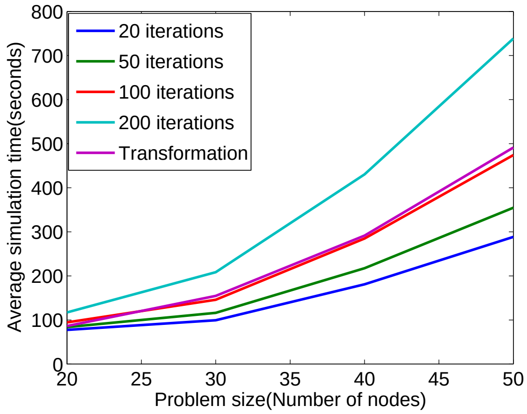


Figure 3. COMPARISON OF AVERAGE SIMULATION TIME USING TRANSFORMATION METHOD AND DIFFERENT ITERATIONS OF PROPOSED ALGORITHM. THESE RESULTS ARE FOR 32 DISCRETIZATIONS OF THE HEADING ANGLE AT EACH TARGET.

bound also converges in terms of the size of the set Θ_d. The average of the percentage of improvement compared with the lower bound after 20 iterations with 32 discretizations is plotted in Figure 2. Clearly, after 20 iterations, the improvement of the lower bound is minimal even after 200 iterations of subgradient optimization. Figure 3 shows the average computation time required for all the algorithms as a function of the size of the problem. Even though the computation time significantly increases as the number of iterations increases, there is not much improvement in the lower bound after 20 iterations. Therefore, one can stop the iterative process after 20 iterations to find a good lower bound within a reasonable amount of time.

Tables 1 and 2 compares the lower bound computed using the proposed method with the lower bound computed from the transformation method [8] for several instances. The first column in the tables indicate the size of an instance of DTSP. Second column refers to the optimal cost corresponding to the Euclidean TSP (which is basically the DTSP without any motion constraints). The lower bounds obtained using the transformation method and the proposed subgradient method are listed in third and fourth columns respectively. The proposed Lagrangian dual algorithm performs better than the transformation method in 28 out of 40 cases with 32 discretizations and in all the 40 cases with 64 discretizations. The transformation method gives a better lower bound in few cases especially when the problem size is small. But as the size of the problem or the number of discretizations of heading angle increases, the proposed algorithm performs better. In few cases, the lower bound from the transformation method is less than the Euclidean TSP and even negative. That is because the transformation method is dependent on selecting an appropriate value for a parameter M as explained in [8]. Our method guarantees a lower bound larger than the bound provided by the Euclidean TSP and could be computed within a reasonable amount of time. Also the proposed method can be generalized to solve any variant of the motion constrained TSP if the Lagrangian relaxation in (16) could be solved.

Table 1. LOWER BOUND COMPARISON WITH 32 DISCRETIZATIONS

Targets	ETSP	Lower bound using transformation	Lower bound using proposed algorithm
20	76.10	66.74	78.78
20	83.56	-44.91	87.79
20	64.44	67.44	66.82
20	86.55	93.83	90.33
20	68.95	54.39	73.18
20	66.91	65.00	69.69
20	75.76	77.54	77.63
20	76.46	84.35	79.47
20	75.30	77.77	78.23
20	84.97	85.97	87.78
30	86.30	82.20	89.72
30	86.94	64.66	91.72
30	78.34	92.61	83.19
30	92.71	81.53	96.85
30	80.53	87.70	84.64
30	84.76	64.49	88.88
30	95.33	93.51	100.19
30	99.22	59.96	104.50
30	88.48	86.49	93.30
30	91.15	84.95	95.25
40	105.05	72.16	108.78
40	99.54	75.66	105.31
40	90.58	106.40	96.52
40	99.93	73.04	105.92
40	108.28	117.20	113.19
40	106.74	104.57	111.70
40	96.98	93.15	100.25
40	100.86	107.94	105.16
40	103.37	71.69	108.04
40	99.71	121.39	106.42
50	114.20	91.66	122.04
50	111.90	127.31	116.45
50	117.42	110.39	122.88
50	117.72	95.69	124.99
50	115.98	120.90	123.61
50	118.90	99.76	125.13
50	119.06	98.18	122.75
50	113.31	124.73	118.04
50	123.73	139.14	129.35
50	113.03	118.58	118.81

Table 2. LOWER BOUND COMPARISON WITH 64 DISCRETIZATIONS

Targets	ETSP	Lower bound using transformation	Lower bound using proposed algorithm
20	74.96	31.73	76.99
20	71.62	20.96	75.10
20	61.38	34.72	64.82
20	66.77	46.05	70.98
20	87.58	77.54	90.65
20	71.22	21.32	74.02
20	73.94	67.47	78.14
20	80.34	59.16	83.63
20	80.19	65.55	83.23
20	81.61	37.23	85.31
30	84.34	69.09	87.92
30	102.99	76.43	107.50
30	92.15	67.39	95.84
30	97.50	-24.09	102.77
30	93.62	53.38	97.41
30	94.45	54.38	98.37
30	85.95	72.83	90.69
30	90.72	66.25	95.13
30	98.83	72.33	101.90
30	90.42	87.01	93.11
40	99.05	1.28	103.47
40	109.69	-28.06	114.76
40	109.80	22.97	114.73
40	102.80	56.35	108.14
40	94.76	-19.25	99.15
40	102.26	89.24	108.58
40	98.60	44.68	103.70
40	104.76	92.01	109.53
40	106.48	99.92	110.99
40	100.38	65.18	104.38
50	110.70	97.58	115.74
50	106.74	77.94	112.09
50	115.05	77.93	120.75
50	105.21	69.02	111.83
50	112.48	102.60	116.90
50	115.25	2.76	122.33
50	123.63	56.82	130.62
50	110.73	-10.59	117.42
50	109.78	101.38	114.54
50	116.28	111.88	120.97

6 CONCLUSIONS

In this paper, we provided a method to compute a lower bound for a vehicle routing problem with motion constraints. This method is explained in detail for a unmanned vehicle modeled as a Dubins car. This approach is quite general and can extended to other routing problems with more general motion constraints. The only requirement for this approach to work is that the corresponding variational problem in (20) is solvable.

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